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Polynomial zeros of the 9-j coefficient

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Abstract. It is shown that there exist polynomial (or non-trivial) zeros for the 9-j coefficient. A simple closed form expression for the polynomial zeros of degree one of the 9-j coefficient is derived from the triple-sum series due to Jucys and Bandzaitis. Polynomial zeros of degree one of the 9-j coefficient are generated from either this closed form expression or from a set of parametric solutions of the multiplicative Diophantine equations: $xyz = uvw$.

1. Introduction

Recently, systematic studies have been made of the polynomial or non-trivial zeros—in particular, zeros of degree one or weight one—of the 3-j and the 6-j coefficients, from the point of view of (i) the embedding of exceptional Lie algebras in orthogonal groups (Van der Jeugt *et al* 1983, Vanden Berghe *et al* 1984, De Meyer and Vanden Berghe 1984); (ii) formal binomial expansions (Srinivasa Rao and Rajeswari 1984); and (iii) multiplicative Diophantine equations (Brudno 1985, Brudno and Louck 1985, Bremner 1986, Bremner and Brudno 1986, Srinivasa Rao and Rajeswari 1986, Srinivasa Rao *et al* 1988b). Here, we discuss for the first time the problem of polynomial zeros of the 9-j coefficient.

We have obtained a simple closed form expression for the polynomial zeros of degree one of the 9-j coefficient and generated them using it. We also generated the inequivalent polynomial zeros of degree one of the 9-j coefficient from a set of parametric solutions to the homogeneous multiplicative Diophantine equations of degree three, namely $xyz = uvw$. However, unlike the single four-parameter solution of $x_1x_2 = u_1u_2$ which generated the complete set of degree-one zeros of the 3-j coefficient (Srinivasa Rao and Rajeswari 1984) and the single eight-parameter solution of $x_1x_2x_3 = u_1u_2u_3$ which generated the complete set of degree-one zeros of the 6-j coefficient (Srinivasa Rao *et al* 1988b), we find that a *set* of solutions of the equation $xyz = uvw$ is necessary to generate the complete set of degree-one zeros of the 9-j coefficient. This complex situation is a direct consequence of the fact that, while single-sum series representations have been obtained by Wigner and Racah for the 3-j and the 6-j coefficients, the 9-j coefficient is represented, at best, by a triple-sum series due to Jucys and Bandzaitis (1977) and Alisaukas and Jucys (1971).

2. Mathematical formulae

The simplest known algebraic form for the 9-j coefficient due to Jucys and Bandzaitis

(1977) is the triple-sum series:

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} = (-1)^{x5} \frac{(dag)(beh)(igh)}{(def)(bac)(icf)}$$
$$\times \sum_{x,y,z} \frac{(-1)^{x+y+z}}{x!y!z!} \frac{(x1-x)!(x2+x)!(x3+x)!}{(x4-x)!(x5-x)!}$$
$$\times \frac{(y1+y)!(y2+y)!}{(y3+y)!(y4-y)!(y5-y)!} \frac{(z1-z)!(z2+z)!}{(z3-z)!(z4-z)!(z5-z)!}$$
$$\times \frac{(p1-y-z)!}{(p2+x+y)!(p3+x+z)!} \tag{1}$$

where

$$\begin{aligned} &0 \leqslant x \leqslant \min(-d+e+f, c+f-i) = XF \\ &0 \leqslant y \leqslant \min(g-h+i, b+e-h) = YF \\ &0 \leqslant z \leqslant \min(a-b+c, a+d-g) = ZF \end{aligned} \tag{2}$$

and

$$\begin{array}{lll} x1 = 2f & x2 = d+e-f & x3 = c+i-f \\ x4 = e+f-d & x5 = c+f-i & y1 = -b+e+h \\ y2 = g+h-i & y3 = 2h+1 & y4 = b+e-h \\ y5 = g-h+i & z1 = 2a & z2 = -a+b+c \\ z3 = a+d+g+1 & z4 = a+d-g & z5 = a-b+c \\ p1 = a+d-h+i & p2 = -b+d-f+h & p3 = -a+b-f+i \end{array} \tag{3}$$

and

$$(abc) = \Delta(abc) \frac{(a+b+c+1)!}{(-a+b+c)!} \tag{4}$$

with

$$\Delta(abc) = [(-a+b+c)!(a-b+c)!(a+b-c)!/(a+b+c+1)!]^{1/2}.$$

If we set $c=0$, the triangular inequalities to be satisfied will lead to $f=i$ and $a=b$, so that the expression for the 9-j coefficient can be shown to reduce to a single-sum series, which corresponds to a 6-j coefficient. The symmetries of the 9-j coefficient will then lead us to the well known special values of this coefficient (Biedenharn and Louck 1981):

$$\begin{Bmatrix} 0 & e & e \\ f & d & b \\ f & c & a \end{Bmatrix} = \begin{Bmatrix} e & 0 & e \\ c & f & a \\ d & f & b \end{Bmatrix} = \begin{Bmatrix} f & f & 0 \\ d & c & e \\ b & a & e \end{Bmatrix} = \begin{Bmatrix} f & b & d \\ 0 & e & e \\ f & a & c \end{Bmatrix}$$
$$= \begin{Bmatrix} a & f & c \\ e & 0 & e \\ b & f & d \end{Bmatrix} = \begin{Bmatrix} b & a & e \\ f & f & 0 \\ d & c & e \end{Bmatrix} = \begin{Bmatrix} e & d & c \\ e & b & a \\ 0 & f & f \end{Bmatrix} = \begin{Bmatrix} c & e & d \\ a & e & b \\ f & 0 & f \end{Bmatrix}$$
$$= \begin{Bmatrix} a & b & e \\ c & d & e \\ f & f & 0 \end{Bmatrix} = \frac{(-1)^{b+c+e+f}}{[(2e+1)(2f+1)]^{1/2}} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}. \tag{5}$$

In the wake of the non-trivial or polynomial zeros of the 6-j coefficient (see, for instance, Srinivasa Rao and Rajeswari 1985) and (5) above, it is obvious that every polynomial zero of the 6-j coefficient would imply a polynomial zero of the 9-j coefficient. The degree of the polynomial zero would be the same in both the cases. However, the 9-j coefficient is a special coefficient with one of the nine angular momenta in it being zero, and such zeros are not the focus of our attention here.

Using the following notation for the Pochammer symbols:

$$(\lambda, k)=\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}=\lambda(\lambda+1)(\lambda+2)\ldots(\lambda+k-1) \qquad k\geqslant 0$$

$$(\lambda, -k)=\frac{(-1)^k}{(1-\lambda, k)} \qquad k<0 \tag{6}$$

equation (1) can be rewritten as

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}=(-1)^{x5}\frac{(dag)(beh)(igh)}{(def)(bac)(icf)}$$
$$\times\frac{\Gamma(1+x1, 1+x2, 1+x3, 1+y1, 1+y2, 1+z1, 1+z2, 1+p1)}{\Gamma(1+x4, 1+x5, 1+y3, 1+y4, 1+y5, 1+z3, 1+z4, 1+z5, 1+p2, 1+p3)}$$
$$\times\sum_{x,y,z}\frac{1}{x!y!z!}\frac{(1+x2, x)(1+x3, x)(-x4, x)(-x5, x)}{(-x1, x)}$$
$$\times\frac{(1+y1, y)(1+y2, y)(-y4, y)(-y5, y)}{(1+y3, y)}$$
$$\times\frac{(1+z2, z)(-z3, z)(-z4, z)(-z5, z)}{(-z1, z)}$$
$$\times\frac{1}{(-p1, y+z)(1+p2, x+y)(1+p3, x+z)} \tag{7}$$

where

$$\Gamma(x, y, \ldots)=\Gamma(x)\Gamma(y)\ldots.$$

The triple-sum series in (7) can now be identified with a triple hypergeometric series. To this end, consider the product of the following ${}_4F_3(1)$, which are generalised hypergeometric functions (Slater 1966) of unit argument:

$${}_4F_3\begin{bmatrix} 1+x2 & 1+x3 & -x4 & -x5; & 1 \\ -x1 & 1+p2 & & 1+p3 & \end{bmatrix}{}_4F_3\begin{bmatrix} 1+y1 & 1+y2 & -y4 & -y5; & 1 \\ 1+y3 & -p1 & & 1+p2 & \end{bmatrix}$$
$$\times{}_4F_3\begin{bmatrix} 1+z2 & -z3 & -z4 & -z5; & 1 \\ -z1 & -p1 & & 1+p3 & \end{bmatrix}$$
$$=\sum_{x,y,z}\frac{1}{x!y!z!}\frac{(1+x2, x)(1+x3, x)(-x4, x)(-x5, x)}{(-x1, x)(1+p2, x)(1+p3, x)}$$
$$\times\frac{(1+y1, y)(1+y2, y)(-y4, y)(-y5, y)}{(1+y3, y)(-p1, y)(1+p2, y)}$$
$$\times\frac{(1+z2, z)(-z3, z)(-z4, z)(-z5, z)}{(-z1, z)(-p1, z)(1+p3, z)}. \tag{8}$$

In equation (8), which represents the product of three independent series, making the following replacements

$$
\begin{array}{lll}
(1+p2, x)(1+p2, y) & \text{by} & (1+p2, x+y) \\
(1+p3, x)(1+p3, z) & \text{by} & (1+p3, x+z) \\
(-p1, y)(-p1, z) & \text{by} & (-p1, y+z)
\end{array}
\tag{9}
$$

enables us to identify the triple series with that in (7). The product of the three ${}_4F_3(1)$ given in (8), with the replacements given in (9), lead us now to a new function in three variables, which can be written as

$$
F^{(3)}\left[\begin{matrix} 1+x2, 1+x3, -x4, -x5; & 1+y1, 1+y2, -y4, -y5; & 1+z2, -z3, -z4, -z5; & 1, 1, 1 \\ -x1; & 1+y3; -z1; & 1+p2, 1+p3, -p1 & \end{matrix}\right] \tag{10}
$$

which is a particular case of an extremely general hypergeometric series defined in three variables by Srivastava (1967), namely

$$
F^{(3)}\left[\begin{matrix} (a)::(b); & (b'); & (b''):(c); & (c'); & (c''); & x, y, z \\ (e)::(f); & (f'); & (f''):(g); & (g'); & (g'') & \end{matrix}\right]
$$

$$
= \sum_{m,n,p} \frac{((a), m+n+p)((b), m+n)((b'), n+p)((b''), p+m)}{((e), m+n+p)((f), m+n)((f'), n+p)((f''), p+m)}
$$

$$
\times \frac{((c), m)((c'), n)((c''), p) x^m y^n z^p}{((g), m)((g'), n)((g'', p) m! n! p!} \tag{11}
$$

where (a) denotes a sequence of parameters (in the notation of Srivastava (1967)), which is an elegant unification of the triple hypergeometric functions of Lauricella (1893), Saran (1954) and Srivastava (1964) functions (Exton 1976). We wish to point out that the *new* generalised hypergeometric function in three variables $\Phi^{(3)}(\alpha_{kl}; \beta_i, \gamma_m; w_k)$, defined by Wu (1973), is the same as $F^{(3)}$ discussed above.

The triple-sum series (1) does not exhibit the 72 symmetries of the 9-j coefficient. In the context of numerically evaluating the 9-j coefficient using (1), it has been shown (Srinivasa Rao *et al* 1988a) that while the (extreme) example

$$
\begin{Bmatrix} 30 & 20 & 10 \\ 30 & 10 & 20 \\ 60 & 30 & 30 \end{Bmatrix}
$$

has $XF+YF+ZF=0$, its symmetries can have $XF+YF+ZF=60$, 80, 100 or 140. Correspondingly, the number of terms to be summed in the triple-sum series (1), reckoned after taking into account the constraints on the ranges of x, y and z placed by $p1$, $p2$ and $p3$ (viz $y+z \leqslant p1$ and if $p2, p3 \geqslant 0$, then $x+y \geqslant |p2|$, $z+x \geqslant |p3|$), for the given 9-j coefficient and its symmetries can have 21, 41, 441, 1681, 9471, 18 081 or 33 761 terms! This is due to the inherent lack of symmetry of (1). On the basis of this observation, we can define the degree of the polynomial zero of the 9-j coefficient as that given by the minimum value of $XF+YF+ZF$ for one or more of its symmetries.

It is to be noted that the conventional single sum over the product of three 6-j coefficients (given in, for example, Biedenharn and Louck (1981)) will not reveal the polynomial zeros of the 9-j coefficient. However, the realisation that the triple-sum series in (1) can be looked upon as a generalised hypergeometric function in three variables, evaluated at unit values for all the variables, enables us to find the polynomial zeros for the 9-j coefficient. All the degree-one polynomial zeros of the 9-j coefficient are then given by the simple closed form expression

$$1-\delta^{\alpha 1,XF,YF,ZF}_{\beta 1,1,0,0}-\delta^{\alpha 2,XF,YF,ZF}_{\beta 2,0,1,0}-\delta^{\alpha 3,XF,YF,ZF}_{\beta 3,0,0,1} \tag{12}$$

where we have introduced the notation:

$$\delta^{a,b,c,d}_{p,q,r,s}=\delta_{a,p}\delta_{b,q}\delta_{c,r}\delta_{d,s} \tag{13}$$

the $\delta_{a,p}$, etc, being the Kronecker delta functions. In (12), the α and β are given by

$$\begin{aligned} \alpha 1 &= (x2+1)(x3+1)x4x5 & \beta 1 &= x1(p2+1)(p3+1) \\ \alpha 2 &= (y1+1)(y2+1)y4y5 & \beta 2 &= (y3+1)p1(p2+1) \\ \alpha 3 &= (z2+1)z3z4z5 & \beta 3 &= z1p1(p3+1) \end{aligned} \tag{14}$$

and the quantities XF, YF and ZF are the upper limits of the summation indices x, y and z given by (2).

3. Multiplicative Diophantine equations

In the study of the polynomial zeros of degree one of the 6-j coefficient, we have shown (Srinivasa Rao *et al* 1988b) that the complete set of zeros can be obtained only from the eight-parameter solution of the multiplicative Diophantine equation: $xyz=uvw$ subject to the condition $z=x+y+u+v+w$. We can also study the polynomial zeros of degree one of the 9-j coefficient from the solutions of the homogeneous multiplicative Diophantine equations of degree three, namely $xyz=uvw$. Since (1) is a triple-sum series (and not a single-sum series as in the case of the 6-j coefficient), the closed form expression (12) for the polynomial zeros of degree one contains four terms and this immediately suggests that the multiplicative Diophantine equations to be solved to generate the degree-one zeros are

$$\alpha 1=\beta 1 \qquad \text{for} \qquad XF=1,\ YF=0,\ ZF=0 \tag{15}$$

$$\alpha 2=\beta 2 \qquad \text{for} \qquad XF=0,\ YF=1,\ ZF=0 \tag{16}$$

$$\alpha 3=\beta 3 \qquad \text{for} \qquad XF=0,\ YF=0,\ ZF=1 \tag{17}$$

where the α and the β are products of three terms given in (14). Furthermore, from (2) it is obvious that $YF=1$ (say) could arise due to $g-h+i=1$ and $b+e-h\geqslant 1$; or $b+e-h=1$ and $g-h+i\geqslant 1$; along with one of $-d+e+f$ or $c+f-i$ being 0; and one of $a-b+c$ or $a+d-g$ being 0. There are therefore eight different cases which should be considered explicitly for each of the above three equations (15), (16) and (17).

Following Bell (1933), we have established (Srinivasa Rao *et al* 1988c) that n^2 parameters are necessary and sufficient to obtain the complete set of solutions of the homogeneous multiplicative Diophantine equation of degree n:

$$x_1x_2\ldots x_n=u_1u_2\ldots u_n \qquad n>1. \tag{18}$$

For the sake of brevity, we discuss here the solutions for one of the three equations. In the case of $YF=1$, if we define n_1, n_2, n_3 to be the products of the row elements of the required nine parameters arranged as a 3×3 array and n_4, n_5, n_6 to be the products of the column elements of the same array, so that

$$n_1n_2n_3=n_4n_5n_6$$

then the solutions for the eight different cases can be grouped into two sets, (I) and (II), of four solutions each:

$$(\mathrm{I})\quad \begin{Bmatrix} a & (-n_1+n_4)/2 & c \\ [(n_2-n_3+2n_5-1)/2]-a & n_1/2 & f \\ (n_2+n_3-1)/2 & (n_4-2)/2 & (-n_2+n_3+n_4-1)/2 \end{Bmatrix}$$

where

$$a=(n_2-n_3+n_5-n_6)\qquad c=b-a\qquad f=i-c \tag{19}$$

$$n_6=n_1\qquad c=b-a\qquad f=d-e \tag{20}$$

$$n_6=n_1\qquad n_5=n_3\qquad f=d-e \tag{21}$$

$$n_5=n_3\qquad c=i-f\qquad f=[(n_1+n_2+n_3-2n_6-1)/2]-a. \tag{22}$$

$$(\mathrm{II})\quad \begin{Bmatrix} a & (-n_1+n_3+n_4-1)/2 & c \\ [(n_2+2n_5-2)/2]-a & (n_1+n_3-1)/2 & f \\ n_2/2 & (n_4-2)/2 & (-n_2+n_4)/2 \end{Bmatrix}$$

where

$$a=(n_2+n_5-n_6-1)/2\qquad c=b-a\qquad f=i-c \tag{23}$$

$$n_6=n_1\qquad c=b-a\qquad f=d-e \tag{24}$$

$$n_6=n_1\qquad n_5=1\qquad f=d-e \tag{25}$$

$$n_5=1\qquad c=i-f\qquad f=(n_1+n_2-n_3-2n_6+1)-a. \tag{26}$$

Of these eight solutions a laborious but straightforward examination reveals the following.

(i) The expressions (19) and (23) are genuine nine-parameter solutions, related by the symmetries of the 9-*j* coefficient *and* the interchange of n_1 by n_2.

(ii) The conditions given in (20), (21), (24) and (25) are inconsistent with the triangular inequalities. We will illustrate this for (25). In this case $g-h+i=1$, $e+f-d=0$, $a+d-g=0$ and hence

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} = \begin{Bmatrix} a & b & c \\ e+f & e & f \\ a+e+f & a+e+f+i-1 & i \end{Bmatrix}.$$

From the triangular inequalities for (beh) and (abc) we have

$$b+e-(a+e+f+i-1)\geqslant 0 \qquad \text{and} \qquad a-b+c\geqslant 0$$

which together imply $c-f-i+1\geqslant 0$, i.e. $c=f+i$ or $f+i-1$ only. Similarly, from the triangular inequalities for (beh) and (cfi):

$$b-a-f-i+1\geqslant 0 \qquad \text{and} \qquad -c+f+i\geqslant 0$$

implying $b-a-c+1\geqslant 0$, i.e. $b=a+c$ or $a+c-1$ only. These restrictions on c and b lead us to the following four cases.

(*a*) $c=f+i$ and $b=a+c$, which imply from (II):

$$c=[(-n_1-n_3+n_4+1)/2]-a \qquad \text{and} \qquad b=(-n_1-n_3+n_4+1)/2.$$

But from (II) we also have $b=(-n_1+n_3+n_4-1)/2$. So, for both the expressions for b to be true, we require $n_3=1$. The conditions given in (25) already require $n_1=n_6$ and $n_5=1$. These, together with $n_3=1$ and the requirement $n_1n_2n_3=n_4n_5n_6$, imply $n_2=n_4$. From (II), when $n_2=n_4$, $i=0$. We have already stated that, when any one of the nine angular momenta is zero, the 9-j coefficient reduces to a 6-j coefficient as in (5) and the zeros of these special 9-j coefficients are a direct consequence of zeros of the 6-j coefficients; and we are not interested in these, since they may be considered as derived polynomial zeros of the 9-j coefficient.

(b) $c=f+i-1$ and $b=a+c$, which imply from (II):

$$c=[(-n_1-n_3+n_4-1)/2]-a \qquad \text{and} \qquad b=(-n_1-n_3+n_4-1)/2.$$

But from (II) we also have $b=(-n_1+n_3+n_4-1)/2$. For both these expressions of b to be true, we require $n_3=0$. Since, by definition, each of the nine parameters in the solution for the multiplicative Diophantine equation take only positive non-zero integral values, we must have strictly $n_3>0$. Thus this case yields no zeros of degree one.

(c) $c=f+i$ and $b=a+c-1=a+f+i-1$. The arguments for case (b) can be repeated and they lead to $n_3=0$.

(d) $c=f+i-1$ and $b=a+c-1=a+f+i-2$, which imply from (II):

$$c=[(-n_1-n_3+n_4-1)/2]-a \qquad \text{and} \qquad b=(-n_1-n_3+n_4-3)/2.$$

Also from (II): $b=(-n_1+n_3+n_4-1)/2$. For both these expressions for b to be true, we require $n_3=-1$, which is forbidden. Thus (25) cannot yield any degree-one zeros of the 9-j coefficient.

(iii) The expressions (22) and (26) are solutions in terms of fewer (than nine) parameters and have one of the angular momenta itself as a free parameter.

Similar results can be obtained for (15) and (17) with the exception that (15) does not yield a full nine-parameter solution. To sum up, we find that of the 24 cases studied, twelve did not yield any degree-one zeros because of inherent inconsistencies and of the remaining twelve studied, only four (two from (16) and two from (17)) are full nine-parameter solutions, the other eight being fewer (than nine) parameter solutions having one of the angular momenta itself as a free parameter.

4. Results and discussion

Polynomial zeros of degree one of the 9-j coefficient were generated on an IBM-PC/AT computer using the closed form expression (12) and the set of twelve solutions of multiplicative Diophantine equations discussed above. Using (12), the polynomial zeros of degree one for all non-zero arguments of the 9-j coefficient that arise when $0<a, b, d, e\leqslant\frac{5}{2}$, were listed on the computer. In this restricted range for the arguments we found 447 polynomial zeros of degree one of the 9-j coefficient. The first twenty of these are given in table 1. Also, this range of arguments contained only three polynomial zeros of degree one of the 6-j coefficient, namely

$$\begin{Bmatrix} 2 & 2 & 2 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{Bmatrix} \qquad \begin{Bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \end{Bmatrix} \qquad \begin{Bmatrix} 4 & \frac{7}{2} & \frac{5}{2} \\ 2 & \frac{5}{2} & \frac{5}{2} \end{Bmatrix}$$

and (5) gives us all the corresponding polynomial zeros of degree one of the 9-j coefficient.

Table 1. The first 20 polynomial zeros of degree one of the 9-j coefficient, with $\sigma = a+b+c+d+e+f+g+h+i$ given in the last column.

a	b	c	d	e	f	g	h	i	σ
0.5	1.0	1.5	1.0	1.5	1.5	1.5	2.5	2.0	13†
0.5	1.0	1.5	1.0	2.0	3.0	1.5	3.0	3.5	17
0.5	1.0	1.5	1.5	0.5	2.0	2.0	1.5	1.5	12†
0.5	1.0	1.5	1.5	2.0	1.5	2.0	3.0	2.0	15†
0.5	1.0	1.5	1.5	2.5	3.0	2.0	3.5	3.5	19
0.5	1.0	1.5	2.0	1.0	2.0	2.5	2.0	1.5	14
0.5	1.0	1.5	2.0	2.5	1.5	2.5	3.5	2.0	17†
0.5	1.0	1.5	2.5	1.5	2.0	3.0	2.5	1.5	16
0.5	1.5	1.0	1.0	1.5	1.5	1.5	2.0	2.5	13†
0.5	1.5	1.0	1.5	1.5	2.0	2.0	2.0	3.0	15†
0.5	1.5	1.0	1.5	2.0	1.5	1.0	2.5	1.5	13†
0.5	1.5	1.0	2.0	1.5	1.5	1.5	2.0	0.5	12†
0.5	1.5	1.0	2.0	1.5	2.5	2.5	2.0	3.5	17†
0.5	1.5	1.0	2.5	1.5	3.0	3.0	2.0	4.0	19
0.5	1.5	2.0	1.0	0.5	1.5	1.5	2.0	1.5	12†
0.5	1.5	2.0	1.0	2.0	2.0	1.5	3.5	3.0	17
0.5	1.5	2.0	1.5	1.0	1.5	2.0	2.5	1.5	14
0.5	1.5	2.0	1.5	2.5	2.0	2.0	4.0	3.0	19
0.5	1.5	2.0	2.0	1.5	1.5	2.5	3.0	1.5	16
0.5	1.5	2.0	2.5	0.5	3.0	3.0	2.0	2.0	17

† These are equivalent ones for a given value of σ.

In table 2 are listed the first few inequivalent polynomial zeros of degree one, for $12 \leqslant \sigma \leqslant 18$. These were generated from the twelve solutions of the multiplicative Diophantine equations discussed earlier in § 3. After generating the polynomial zeros, the results were further analysed with the help of a program by which the inequivalent 9-j coefficients were isolated (by dropping the equivalent ones, which are symmetries of the listed one). The significant point to be noted is that the nine-parameter solutions do not generate all these listed zeros. This is obvious by looking at (12), since the nine-parameter solutions are from (16) for (17) giving rise to the third or the fourth term of (12) being equal to 1. So, the set of twelve solutions of the multiplicative Diophantine equations is necessary to generate all the polynomial zeros of degree one of the 9-j coefficient. A scan of the tables of 9-j coefficients (Jahn and Howell 1959) reveals a listing of 67 polynomial zeros and of these, 60 are zeros of degree one. Table 2 lists all the inequivalent zeros of degree one for $\sigma \leqslant 18$.

Finally, while the closed form expressions, or the solutions of single multiplicative Diophantine equations, generate all the polynomial zeros of degree one of the 3-j and the 6-j coefficients, *all* the polynomial zeros of degree one of the 9-j coefficient arise from either the closed form expression (12) or a *set* of solutions of multiplicative Diophantine equations (and there exists no single solution which will generate them all). Therefore, the generation of all the polynomial zeros of degree one of the 9-j coefficient from the closed form expression (12) is straightforward, simpler and economical. These polynomial zeros imply that certain specific reduced matrix elements of the tensor product of two irreducible tensors taken between certain specific well defined angular momentum states (de Shalit and Talmi 1963) are zero. It is necessary to investigate the physical significance of these vanishing matrix elements in quantum-mechanical studies.

Table 2. All the inequivalent polynomial zeros of degree one of the 9-j coefficient for $\sigma \leq 18$. The last three columns represent which of the 12 solutions of the multiplicative Diophantine equations give rise to the entries in this table. Column P represents the four nine-parameter solutions, column Q the four solutions of (15) and column R represents the remaining four solutions. Y stands for yes and N for no.

a	b	c	d	e	f	g	h	i	σ	x	y	z	P	Q	R
2.0	1.5	0.5	1.5	0.5	1.0	1.5	2.0	1.5	12	0	0	1	Y	Y	Y
1.5	2.5	1.0	1.0	1.5	0.5	1.5	2.0	1.5	13	0	1	0	Y	Y	Y
2.0	2.5	0.5	2.0	1.5	1.5	1.0	2.0	1.0	14	1	0	0	Y	Y	Y
2.0	1.5	0.5	1.5	1.0	1.5	1.5	2.5	2.0	14	0	0	1	Y	Y	Y
2.0	3.0	1.0	2.0	0.5	1.5	1.5	2.0	1.5	15	0	1	0	Y	Y	Y
2.5	3.0	0.5	2.0	1.5	1.5	1.5	2.5	1.0	16	1	0	0	Y	Y	Y
2.0	3.0	1.0	2.0	1.5	1.5	1.0	2.5	1.5	16	1	0	0	N	Y	Y
2.0	2.5	0.5	1.5	1.5	2.0	1.5	3.0	1.5	16	1	0	0	Y	Y	Y
2.0	1.5	0.5	2.5	1.0	1.5	2.5	2.5	2.5	16	0	0	1	Y	Y	Y
3.0	2.5	0.5	2.0	0.5	1.5	2.0	3.0	2.0	17	0	0	1	Y	Y	Y
3.0	2.0	1.0	1.5	1.0	0.5	3.5	3.0	1.5	17	0	0	1	Y	Y	Y
2.5	2.0	1.5	1.0	0.5	1.5	3.5	2.5	2.0	17	1	0	0	Y	Y	Y
2.0	2.0	1.0	2.0	1.5	0.5	3.0	3.5	1.5	17	0	0	1	Y	Y	Y
2.0	1.5	1.5	1.5	1.5	1.0	3.5	2.0	2.5	17	0	1	0	N	Y	Y
3.0	2.5	0.5	2.0	1.0	1.0	3.0	3.5	1.5	18	0	0	1	Y	Y	Y
2.5	3.5	0.5	1.5	2.0	1.5	2.0	3.0	2.0	18	0	1	0	Y	N	N
2.5	3.0	0.5	1.5	1.5	2.0	2.0	3.5	1.5	18	1	0	0	Y	Y	Y
2.0	3.0	2.0	1.5	0.5	1.0	1.5	3.5	3.0	18	0	0	1	Y	Y	Y
2.0	3.0	1.0	1.5	1.5	2.0	1.5	3.5	2.0	18	1	0	0	N	Y	Y
2.0	2.5	1.5	1.5	1.0	1.5	1.5	3.5	3.0	18	0	0	1	N	Y	Y
2.0	2.5	0.5	2.5	2.5	2.0	1.5	3.0	1.5	18	1	0	0	N	Y	Y
2.0	2.5	0.5	1.5	2.5	2.0	1.5	3.0	2.5	18	0	1	0	Y	N	N
2.0	2.0	1.0	2.5	1.0	1.5	2.5	3.5	2.5	18	0	0	1	Y	Y	Y

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